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Contractions, cycle double covers, and cyclic colorings in locally connected graphs

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Abstract

A finite, undirected graph is called *locally connected*, if the neighborhood of every vertex induces a connected subgraph. In this paper we study the existence of edges in locally connected k-connected graphs whose contraction keeps the graph locally connected k-connected.

As an application, we prove that the statement of the famous cycle double cover conjecture is true for locally connected graphs.

Moreover, we prove that a conjecture of Plummer and Toft on cyclic colorings of 3-connected planar graphs holds when restricted to locally connected graphs.

Keywords: Local connectivity; Connectivity; Contraction; Cycle double cover; Cyclic chromatic number; Pancyclicity; Square of a graph; Lexicographic product

1. Introduction

An edge in a finite, undirected simple *graph* G is k*-contractible* if the graph G/e obtained from G by *contracting* e, i.e. identifying the vertices of e and eliminating multiple edges, is k-connected.

The knowledge on the distribution of k-contractible edges in k-connected graphs has certainly an appeal to be useful for inductive proofs. Indeed, the existence of a 3-contractible edge in any 3-connected graph nonisomorphic to K_4 whose contraction preserves 3-connectivity has been used in [25] to give a short proof for the existence of convex straight line embeddings for 3-connected planar graphs. Another application, which uses a more sophisticated result on the

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local distribution of 3-contractible edges, has been one of the main tools for proving a result on cyclic colorings in 3-connected graphs [19]. Furthermore, the existence of a k-contractible edge in a triangle free $(k+3)$-connected graph led to the existence of an induced cycle C in every $(k+3)$-connected graph G such that $G - V(C)$ is k-connected [24].

The distribution of k-contractible edges is subject of lasting studies, which seems to contradict a certain lack of further applications. The reason for this could be that elementary graph properties which are invariant under contraction of single edges are rare as well: Almost none of the "basic" graph invariants like maximum and minimum degree or chromatic number is preserved by contracting edges or is at least monotone under this operation. (Embeddability into a given surface is maintained by contracting edges, and, more generally, any graph class described by forbidden minors is closed under contraction by definition. Hence the *genus* of G is an example of a graph invariant which does not increase by contracting edges.)

Another reason for the lack of applications in case of *higher connectivity* within this area is certainly the fact that for every $k \geqslant 4$ the class of graphs which do not admit k-contractible edges at all is infinite (see [12]). One could still hope for "good" characterizations of these classes, but already the class of 4-connected graphs without 4-contractible edges is rich, which is reflected by the observation that arbitrary large complete minors may occur here (and, more specifically, by the stronger statement that most of these graphs are line graphs of *cubic cyclically* 4-*edge-connected graphs*, a class of graphs which is far from being well understood).

The aim of this paper is to show that *locally connected* k-connected graphs have similar contractibility properties as k-connected graphs in general. In particular, we show that every vertex in a locally connected 2-connected graph is incident with at least two edges whose contraction preserves local connectivity (and, thus, 2-connectivity up to trivial cases). We obtain local properties of vertices in locally connected 3-connected graphs which are not incident with a 3-contractible edge whose contraction preserves also *local connectivity*, and we use the latter result to prove that every locally connected 3-connected graph nonisomorphic to K_4 admits a 3-contractible edge, whose contraction maintains local connectivity, too.

The considerations extend to 4-connected graphs: Every locally connected 4-connected graph contains a 4-contractible edge whose contraction preserves local connectivity, unless the graph belongs to an infinite class of exceptional graphs—but in contrast to the above mentioned result on arbitrary 4-connected graphs, the exceptional class is very simple: it consists of the squares of cycles of length at least 5. To prove this result, we prove that if a 4-connected (not necessarily locally connected) graph of minimum degree 4 has a 4-contractible edge then there must be a 4-contractible edge incident with a vertex of degree 4.

For higher connectivity, we construct a class of locally connected 5-regular essentially 6-connected graphs which cannot be contracted to any 5-connected graph by contracting less than four edges.

As an application of these results, we prove that the statement of the famous cycle double cover conjecture [22,23] holds for locally connected graphs.

Moreover, we sketch a proof for the statement that every 3-connected locally connected planar graph without facial cycles of length exceeding k admits a vertex coloring with at most $k+2$ colors such that no color occurs twice on the same facial cycle. This has been conjectured for 3-connected planar graphs in general by Plummer and Toft [19].

Finally, we touch upon open problems concerned with locally connected graphs.

2. Preserving local connectivity by contraction

Let us start with the question under which conditions contracting an edge preserves local connectivity. For terms and notation not defined here we refer to [1] or [4]. For a subset X of the vertex set $V(G)$ of a graph G, let us denote its *neighborhood* by $N_G(X) := \{y \in V(G) - X: xy \in E(G) \text{ for some } x \in X\}$. For a single vertex x, we write for short $N_G(x) := N_G(\{x\})$. By $G(X) := (X, \{xy \in E(G): x, y \in X\})$ we denote the subgraph *induced* by X in G.

Lemma 1. *If xy is an edge of a graph G such that $G(N_G(x))$, $G(N_G(y))$, and $G(N_G(y)) - x$ are connected then $G(N_G(\{x, y\}))$ is connected.*

Proof. Let $H := G(N_G(\{x, y\}))$. We may assume that there exists a vertex $a \in N_G(y) - \{x\}$, for if, otherwise, $N_G(y) = \{x\}$ then y is isolated in the connected graph $G(N_G(x))$, implying $N_G(x) = \{y\}$; so H would be the empty graph, which is trivially connected. Since $G(N_G(y)) - x \subseteq H$ is connected, for every $b \in N_G(y) - \{x\}$ there exists a b, a-path in H. If $c \in N_G(x) - \{y\}$ then there exists a nontrivial c, y-path P in $G(N_G(x))$. $P - \{y\}$ is a c, b-path for some vertex $b \in N_G(y)$ contained in H, and, thus, there exists a c, a-path in H. Hence for all $d \in V(H)$, there exists a d, a-path in H. □

Let xy be an edge of a graph G. We say that the graph G/xy is obtained from G by *contracting xy (to a new vertex w)*, if $V(G/xy) = (V(G) - \{x, y\}) \cup \{w\}$, where $w \notin V(G)$, and $E(G/xy) = E(G - \{x, y\}) \cup \{wz: z \in N_G(\{x, y\})\}$; an edge $uv \in E(G)$ and an edge $u'v' \in E(G/xy)$ *correspond* to each other, if either $uv = u'v'$ or $u \in \{x, y\} \wedge u' = w \wedge v' = v (\in N_G(\{x, y\}))$.

Theorem 1. *If xy is an edge of a locally connected graph G and $G(N_G(x)) - y$ is connected or $G(N_G(y)) - x$ is connected then G/xy is locally connected.*

Proof. Let G' be the graph obtained from G by contracting xy to a single vertex w. If one of $G(N_G(x)) - y$, $G(N_G(y)) - x$ is connected then Lemma 1 applies either directly or to y, x for x, y, proving that $G'(N_{G'}(w)) = G(N_G(\{x, y\}))$ is connected. Now consider some $z \in V(G') - \{w\} \subseteq V(G)$. If $z \notin N_G(w)$ then $G'(N_{G'}(z)) = G(N_G(z))$ is connected trivially. If, otherwise, z is adjacent to one of x, y then $G'(N_{G'}(z))$ is obtained from the connected graph $G(N_G(z) \cup \{x, y\})$ by contracting xy to w. It follows that $G'(N_{G'}(z))$ is connected for all $z \in V(G')$. □

The condition to x, y in Theorem 1 is sufficient but not necessary for G/xy being locally connected, as the graph in Fig. 1 shows, where the two vertices of degree 6 play the role of x, y.

Note that Theorem 1 is in fact a theorem on locally connected 2-*connected* graphs, since every connected locally connected graph on at least 3 vertices is 2-connected as well (this generalizes

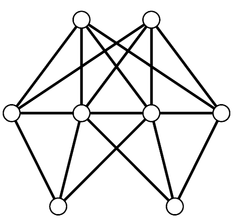

Fig. 1. Illustration for Theorem 1.

to higher local connectivity, as it has been proved in [3], see also the last section). We obtain the following corollary.

Corollary 1. *Every vertex of a locally connected* 2*-connected graph G is incident with two edges such that the graph obtained from G by contracting either of them is locally connected* 2*-connected.*

Proof. For $x \in V(G)$, the graph $G(N_G(x))$ has order at least 2 and, thus, admits a spanning tree with at least two noncutvertices y, y'. So Theorem 1 applies to xy and to xy' for xy. $\square$

3. Locally connected 3-connected graphs

In order to prove the results of the next two sections we need further concepts from graph connectivity theory. For a noncomplete graph, let $\kappa(G) := \min\{|T|\colon\ T \subseteq V(G),\ G - T$ is disconnected$\}$ be its connectivity. This extends to *complete graphs K_n* on n vertices by defining $\kappa(K_n) := \min\{n-1, 1\}$. Let $\mathcal{T}(G) := \{T \subseteq V(G)\colon\ |T| = \kappa(G),\ G - T$ is disconnected$\}$ denote the set of *smallest separators* of G. We say that $X \subseteq V(G)$ *separates* $Y \subseteq V(G)$ if Y intersects two distinct components of $G - X$. It is easy to see that $T \in \mathcal{T}(G)$ separates $T' \in \mathcal{T}(G)$ if and only if T' separates T, which is in turn equivalent to the fact that T intersects *every* component of $G - T'$.

Consider a set $\mathcal{S} \subseteq \mathcal{P}(V(G))$ of vertex sets in G and let $\mathcal{T}_{\mathcal{S}}(G) := \{T \in \mathcal{T}(G)\colon\ S \subseteq T$ for some $S \in \mathcal{S}\}$. For $T \in \mathcal{T}_{\mathcal{S}}(G)$, the union of the vertex sets of at least one but not of all components of $G - T$ is called a *$T - \mathcal{S}$-fragment*; an *$\mathcal{S}$-fragment* is a $T - \mathcal{S}$-fragment for some $T \in \mathcal{T}_{\mathcal{S}}$. Given a $T - \mathcal{S}$-fragment F, T is uniquely determined by $T = N_G(F)$; for an $\mathcal{S}$-fragment F we shall frequently use T_F instead of $N_G(F)$. Moreover, $\overline{F} := V(G) - (F \cup N_G(F))$ is an $\mathcal{S}$-fragment, too, and $\overline{\overline{F}} = F$. An *$\mathcal{S}$-end* is an inclusion minimal $\mathcal{S}$-fragment; an $\mathcal{S}$-end B will also be called a *$T_B - \mathcal{S}$-end*. An *$\mathcal{S}$-atom* A is an $\mathcal{S}$-fragment of minimum cardinality; an $\mathcal{S}$-atom A will also be called a *$T_A - \mathcal{S}$-atom*.

If $\mathcal{S} = \{\emptyset\}$ then the symbol $\mathcal{S}$ is left out, thus defining the concepts of *T-fragments*, *fragments*, *ends*, and *T-ends*.

Fragments have the following fundamental property: If F is a T-fragment and F' is a T'-fragment such that $F \cap F' \neq \emptyset$ then

$$|F \cap T'| \geqslant |\overline{F'} \cap T|, \tag{1}$$

and if equality holds in (1) then $F \cap F'$ is a $T_G(F, F')$-fragment, where $T_G(F, F') = (F \cap N_G(F')) \cup (N_G(F) \cap N_G(F')) \cup (N_G(F) \cap F')$. (For a proof see, for example, [14].)

Suppose that F is an $\mathcal{S}$-end and that there exists an $S' \in \mathcal{S}$ and a $T' \in \mathcal{T}(G)$ such that $S' \subseteq T' - \overline{F}$ and $T' \cap F \neq \emptyset$. For this setup, we mention some important consequences (see also [14, Lemma 1]).

If F' is any T'-fragment then $S' \subseteq T_G(F, F')$ holds, and $F \cap F' \subsetneqq F$ cannot be an $\mathcal{S}$-fragment. If F' intersects F then, by (1), $|F \cap T'| > |\overline{F'} \cap T|$, and $|F' \cap T| > |\overline{F} \cap T'| \geqslant 0$; it follows $\overline{F} \cap \overline{F'} = \emptyset$, for otherwise (1) applied to $\overline{F}, \overline{F'}$ instead of F, F' would cause a contradiction. If F' does not intersect F then $|F' \cap T| > 0$ holds as well, since every vertex in $T' \cap F$ must have a neighbor in $F' \cap (F \cup T)$.

To summarize this, for every T'-fragment F', one of $F \cap F'$, $\overline{F} \cap \overline{F'}$ is empty, and F' intersects T. In particular, T' separates T and, thus, T separates T'.

Applications of (1) or its immediate consequences are indicated in the text by $\geqslant^*$, $>^*$, $\subseteq^*$, etc.

According to [14], a graph G is called *$\mathcal{S}$-critical*, if

(C1) $\mathcal{S} \neq \emptyset$;
(C2) for every $S \in \mathcal{S}$, there exists a $T \in \mathcal{T}(G)$ with $S \subseteq T$; and
(C3) for every $\mathcal{S}$-fragment F there exists an $S' \in \mathcal{S}$ and a $T' \in \mathcal{T}(G)$ such that $S' \subseteq T' - \overline{F}$ and $T' \cap F \neq \emptyset$.

Condition (C3) can be considered as a density condition to $\mathcal{T}_{\mathcal{S}}(G)$. In most of the specializations, for every $\mathcal{S}$-fragment F there will be an $S' \in \mathcal{S}$ such that $S' \cap \overline{F} = \emptyset$ and $S' \cap F \neq \emptyset$ (this could be considered as a density condition to $\mathcal{S}$). If $\mathcal{S}$ has this property, then (C2) implies (C3), so in order to prove that G is $\mathcal{S}$-critical it suffices to check (C1) and (C2).

As an example, consider a graph G of connectivity $k \geqslant \ell \geqslant 0$ and $\mathcal{S}_\ell := \{S \subseteq V(G)\colon |S| \leqslant \ell\}$. For every fragment F, there is an $S' \in \mathcal{S}_\ell$ intersecting F but not intersecting $\overline{F}$. Therefore, a noncomplete graph is $\mathcal{S}_\ell$-critical if and only if for every $S \subseteq V(G)$ with $|S| \leqslant \ell$ there exists a $T \in \mathcal{T}(G)$ with $S \subseteq T$. Noncomplete $\mathcal{S}_\ell$-critical graphs of connectivity k are often called *(k, ℓ)-graphs* or *ℓ-critically k-connected graphs*. Hence $\mathcal{S}$-criticality generalizes, in particular, the concept of critically k-connected graphs (where $\ell = 1$).

Another example is constituted by setting $\mathcal{R} := \{\{x, y\}\colon xy \in E(G)\}$ for some noncomplete graph G of connectivity k. It is easy to see that an edge e is not k-contractible in G if and only if there exists a $T \in \mathcal{T}(G)$ with $V(e) \subseteq T$. From this it follows straightforward that G has no k-contractible edge if and only if it is $\mathcal{R}$-critical. Such a graph is sometimes called *contraction-critically k-connected*.

More examples as well as further properties of (generalized) fragments, ends, and atoms have been discussed in [14].

Let us come back to locally connected graphs. To emphasize it: The property of an edge e in some locally connected k-connected graph G to be k-contractible means that G/e is k-connected. It does *not* mean that G/e is locally connected k-connected. We start with the following lemma.

Lemma 2. *Let xy be an edge of a locally connected 3-connected graph G such that $G(N_G(x)) - y$ is connected and $\{x, y, z\}$ separates G for some vertex z.*

Then every $\{x, y, z\}$-fragment contains a vertex $y' \in N_G(x)$ such that the graph $G(N_G(x)) - y'$ is connected.

Proof. Set $H := G(N_G(x))$ and let S be a spanning tree of H. Consider a $T := \{x, y, z\}$-fragment F. Let us assume, to the contrary, that $H - y'$ is disconnected for every $y' \in V(H) \cap F$. Since $x \in T \in \mathcal{T}(G)$, there exists a $y' \in F \cap V(H)$. Since every component of $H - y'$ contains an end vertex of S, which does not separate H and is, therefore, in $\overline{F} \cup T$ by assumption, every component of $H - y'$ contains a vertex of T, so one of y, z. It follows that $H - y'$ has precisely two components $C, \overline{C}$, where we may assume that $y \in V(C)$ and $z \in V(\overline{C})$. In particular, $yz \notin E(G)$. If $V(C)$ intersected $\overline{F}$ then $H - y$ would be disconnected, contradicting the condition to y. So $V(C)$ does not intersect $\overline{F}$, and, in particular, y has no neighbor in $\overline{F} \cap V(H)$. Since G is 3-connected, y has neighbors $a \in F$ and $b \in \overline{F}$, and since G is locally connected there must be an a, b-path P in $G(N_G(y))$. P must intersect T, so it must contain x or z. It cannot contain x since y has no neighbor in $\overline{F} \cap V(H)$, so it must contain z. But then z is adjacent to y, contradicting $yz \notin E(G)$. $\square$

Lemma 2 has already an interesting consequence for the existence of 3-contractible edges whose contraction preserves also local connectivity.

Corollary 2. *Suppose that x is a vertex of a locally connected 3-connected graph G such that $G(N_G(x))$ is a path in G, and let y be either of its two end vertices.*

Then G/xy is locally connected 3-connected.

Proof. Since $G(N_G(x)) - y$ is connected, G/xy is locally connected by Theorem 1. Assume to the contrary, that G/xy is not 3-connected. Then $T := \{x, y, z\}$ forms a separator for some vertex z. By Lemma 2, each T-fragment contains a $y' \in N_G(x)$ such that $G(N_G(x)) - y'$ is connected. So there are at least three vertices y' in $N_G(x)$ such that $G(N_G(x)) - y'$ is connected, which is impossible since $G(N_G(x))$ is a path. □

Lemma 3. *Suppose that x is a vertex of a locally connected 3-connected graph such that contracting any edge incident with x produces a graph which is not locally connected 3-connected.*

Let $\mathcal{S} := \{\{x, y\}\colon\ y \in N_G(x),\ G(N_G(x)) - y$ is connected$\}$.

Then G is $\mathcal{S}$-critical, and every $\mathcal{S}$-end has cardinality 1.

Proof. Let $H := G(N_G(x))$. Let y be an end vertex of a spanning tree of H. Clearly, $H - y$ is connected, so $\{x, y\} \in \mathcal{S}$ and, thus, $\mathcal{S} \neq \emptyset$.

Let $\{x, y\} \in \mathcal{S}$. By Theorem 1, G/xy is locally connected. It follows that G/xy is not 3-connected, so there exists a $z \in V(G)$ such that $\{x, y, z\}$ separates G. In order to prove that G is $\mathcal{S}$-critical, it thus suffices to prove that every $\mathcal{S}$-fragment F contains some $y' \in N_G(x)$ such that $\{x, y'\} \in \mathcal{S}$ (then $\{x, y'\} \subseteq T' - \overline{F}$ for some $T' \in \mathcal{T}(G)$, and $T' \cap F \neq \emptyset$). This follows, however, directly from Lemma 2.

For the second part, suppose that F is a $T := \{x, y, z\} - \mathcal{S}$-end, where $\{x, y\} \in \mathcal{S}$. It contains a $y' \in N_G(x)$ such that $H - y'$ is connected, and there exists a $T' \in \mathcal{T}(G)$ containing x, y'. Since F is an $\mathcal{S}$-end, T' separates T. Hence T' intersects $\overline{F}$, too, so $T' = \{x, y', z'\}$ for some $z' \in \overline{F}$. Now if $F' \cap F \neq \emptyset$ for some T'-fragment F', then $T_G(F, F')$ is one of $\{x, y', y\}, \{x, y', z\}$, implying that $F \cap F'$ would be an $\mathcal{S}$-fragment properly contained in F, a contradiction. So $F = \{y'\}$, proving the lemma. □

The following result indicates that contracting edges may serve as an induction tool in the class of locally connected 3-connected graphs.

Theorem 2. *For every vertex x in a locally connected 3-connected graph G nonisomorphic to K_4 there exists an edge e such that $V(e)$ has distance at most 1 from x and G/e is locally connected 3-connected.*

Proof. Suppose that the statement is false for some x. As in Lemma 3, let $\mathcal{S} := \{\{x, y\}\colon\ y \in N_G(x),\ G(N_G(x)) - y$ is connected$\}$. Then G is $\mathcal{S}$-critical, and there exists an $\mathcal{S}$-end consisting of a single vertex y'. Its neighborhood admits a spanning path abc, and contracting ay' produces a locally connected graph G'. By assumption, G' is not 3-connected, implying that there exists a $T' \in \mathcal{T}(G)$ containing a, y'. But T' cannot separate $N_G(y')$, a contradiction. □

This reduction theorem leads immediately to a *construction method* for locally connected 3-connected graphs. Let w be a vertex of an arbitrary graph G and let $X, Y \subseteq N_G(w)$ such that $X \cup Y = N_G(w)$. We say that G' is obtained from G by *splitting* w (*into* (x, y) *according to* (X, Y)), if $V(G') = (V(G) - \{w\}) \cup \{x, y\}$ (where $x \neq y$ are not in $V(G)$) and $E(G') = E(G - w) \cup \{xy\} \cup \{xz\colon z \in X\} \cup \{yz\colon z \in Y\}$. Note that G can be obtained from G' by contracting xy to w. We call the splitting *proper* if $X \cap Y = \emptyset$ and we call it *admissible* if $|X|, |Y|, |X \cup Y| - 1 \geqslant 2$. Note that a proper splitting with $|X|, |Y| \geqslant 2$ is already admissible and will keep a 3-connected graph being 3-connected.

By Tutte's celebrated *Wheel theorem*, every 3-connected graph can be obtained from a wheel by performing edge additions and proper admissible splittings. An analogue of this theorem is far from being true in the locally connected case, since the addition of a single edge to a locally connected graph could produce a graph which is not locally connected, and, even worse, *any* proper splitting will result in a graph which is not locally connected. (Even performing a non-proper admissible splitting could produce a 3-connected graph which is not locally connected.) A straightforward but artificial way to overcome these problems is to call a splitting as above *locally connected* if $G'(N_{G'}(z))$ is locally connected for every $z \in X \cup Y \cup \{x, y\}$. Then, by Theorem 2, every locally connected 3-connected graph can be obtained from a graph K_4 by performing locally connected admissible splittings.

4. A two-paths-theorem

It has been proved in [13] that removing a vertex x of a 3-connected graph not incident with a 3-contractible edge produces either a cycle or a graph G with two induced paths P_1, P_2 of length at least 1 whose vertices have degree 2 such that $G - V(P_1)$, $G - V(P_2)$ are 2-connected and $V(P_1) \cap (V(P_2) \cup N_G(V(P_2))) = \emptyset$, i.e. the distance of $V(P_1)$ and $V(P_2)$ in G is at least 2. This property turned out to be useful in various contexts, as it is demonstrated in [12]. Here we will prove an analogous statement for the locally connected case.

In the following theorem we fix some vertex x and look at some selected edges xy incident with x; for such an edge xy, G/xy would be *locally connected* by Theorem 1. If, among these edges, there was a 3-contractible one then we could reduce G by contraction to a smaller locally connected 3-connected graph. If this reduction is not possible, we obtain some structural information about $G - x$.

Theorem 3. *Let x be a vertex of a locally connected* 3*-connected graph G. Suppose that every edge xy incident with x such that $G(N_G(x)) - y$ is connected is not* 3*-contractible.*

Then either $G - x$ is a cycle, or $G - x$ admits two induced paths P_1, P_2 of length at least 1 *with the following properties*:

(1) $d_{G-x}(y) = 2$ *and $G(N_G(x)) - y$ is connected for every $y \in V(P_1) \cup V(P_2)$*;
(2) $N_{G-x}(V(P_1)) \cup N_{G-x}(V(P_2)) \subseteq N_G(x)$;
(3) $V(P_1) \cap (V(P_2) \cup N_{G-x}(V(P_2))) = \emptyset$ (*and, consequently,* $V(P_2) \cap (V(P_1) \cup N_{G-x}(V(P_1))) = \emptyset$) (*so $V(P_1)$, $V(P_2)$ have distance at least* 2 *in $G - x$*);
(4) $(G - x) - V(P_1)$ *and* $(G - x) - V(P_2)$ *are* 2*-connected*;
(5) *the graph obtained from G by contracting any edge of $G - x$ incident with some vertex of $V(P_1) \cup V(P_2)$ is locally connected* 3*-connected*; *and*
(6) $G - V(P_1)$ *and* $G - V(P_2)$ *are locally connected* 3*-connected.*

Proof. Let $\mathcal{S} := \{\{x, y\}\colon\ y \in N_G(x),\ G(N_G(x)) - y \text{ is connected}\}$ as in Lemma 3. Assume $G - x$ is not a cycle. Let $\mathcal{E} := \{\{x, y\} \in \mathcal{S}\colon\ \{y\} \text{ is an } \mathcal{S}\text{-end}\}$.

Claim 1. *G is $\mathcal{E}$-critical.*

Since there exist $\mathcal{S}$-ends and since each of them consists of a single vertex y such that $G(N_G(x)) - y$ is connected by Lemma 3, $\mathcal{E} \neq \emptyset$ holds. Since G is $\mathcal{S}$-critical, every $\{x, y\}$ in $\mathcal{E} \subseteq \mathcal{S}$ is contained in some $T \in \mathcal{T}(G)$. Finally, consider any $\mathcal{E}$-fragment F; since F is an $\mathcal{S}$-fragment as well, it must contain an $\mathcal{S}$-end, which consists again of a single vertex y such that $S' := \{x, y\}$ is in $\mathcal{S}$. Since S' is contained in some $T' \in \mathcal{T}(G)$, Claim 1 follows.

Claim 2. *Every $\mathcal{E}$-end has cardinality* 1.

This follows just as above: Take a $T - \mathcal{E}$-end F. It contains an $\mathcal{S}$-end consisting of a single vertex y such that $S' := \{x, y\}$ is in $\mathcal{S}$, and S' is in turn contained in some $T' \in \mathcal{T}(G)$. Since T' separates T, T' intersects $\overline{F}$, too, and T intersects every T'-fragment in precisely one vertex. Now $F \cap T' = \{y\}$, and if F' is a T'-fragment then $F' \cap F = \emptyset$, for otherwise $\{x, y\} \subseteq T_G(F, F') \in \mathcal{T}(G)$ would follow, implying that $F' \cap F$ would be an $\mathcal{E}$-fragment properly contained in the $\mathcal{E}$-end F. Consequently, $F = F \cap T' = \{y\}$, proving Claim 2.

Claim 3. *There exist two induced paths P_1, P_2 of length at least* 1 *satisfying conditions* (1)–(3) *of the statement.*

Let $\{y_1\}$ be an arbitrary $\mathcal{E}$-end and $\{z_1\}$ be an $\mathcal{S}$-end contained in $N_G(y_1)$. Note that y_1, z_1 form a path of length 1 whose vertices have degree 2 in $G - x$. We choose P_1 among all induced paths of $G - x$ consisting of vertices of degree 2 and containing y_1, z_1 in such a way that P_1 is as long as possible. Since $G - x$ is not a cycle and 2-connected, $N_{G-x}(V(P_1))$ forms a separating set in $G - x$ consisting of two distinct vertices s, t of degree exceeding 2 in $G - x$. Since neither y_1 nor its neighbor z_1 separates $G(N_G(x))$, there exists a cycle C in $G(N_G(x))$ containing $y_1 z_1$. Observe that $V(C)$ must contain $V(P_1) \cup \{s, t\}$. In particular, $s, t \in N_G(x)$ and $N_G(x) - y$ is connected for *every* $y \in V(P_1)$. Since every vertex in P_1 has degree 3 in G and is adjacent to a vertex of degree 3 which does not separate $G(N_G(x))$, P_1 is formed by certain $\mathcal{E}$-ends. Without loss of generality, we may assume that y_1 is an end vertex of the path P_1, so $P_1 = y_1, z_1, \ldots,$ and we may assume that s is the neighbor of y_1 among s, t.

Now $F := V(P_1) - \{y_1\}$ forms an $\{x, y_1, t\} - \mathcal{E}$-fragment in G, and $\overline{F}$ contains an $\mathcal{E}$-end, which therefore consists of a single vertex y_2 having degree 3 in G; its neighborhood contains an $\mathcal{S}$-end consisting of a single vertex z_2, and as above we take a path P_2 among all induced paths of $G - x$ consisting of vertices of degree 2 and containing y_2, z_2 in such a way that P_2 is as long as possible. Since s, t have degree exceeding 2 in $G - x$, $V(P_2)$ does not intersect $V(P_1) \cup \{s, t\}$.

This proves Claim 3.

Let us call an induced subpath P of $G - x$ of length at least 1 a *link*, if $d_{G-x}(y) = 2$ and $G(N_G(x)) - y$ is connected for every $y \in V(P)$, and all neighbors of P in $G - x$ have degree exceeding 2 in $G - x$ and are contained in $N_G(x)$. Note that $V(P) \cap (V(P') \cup N_{G-x}(V(P'))) = \emptyset$ for any two distinct links P, P'. The paths P_1, P_2 of Claim 3 might fail to be links, as some their neighbors in $G - x$ could have degree 2 in $G - x$.

We prove now that there exist paths P_1, P_2 satisfying (1)–(4) by induction on $|V(G)|$. By Claim 3, $|V(G)| \geqslant 6$, and if $|V(G)| = 6$ then P_1, P_2 are paths of length 1 and $N_{G-x}(V(P_1)) = N_{G-x}(V(P_2)) = \{a, b\}$, where a, b are adjacent (as $G - x$ is not a cycle). In this case, P_1, P_2 satisfy (4), too.

We may assume that there exists a link P of length at least 1 such that $(G-x)-V(P)$ is not 2-connected, for otherwise we take paths P_1, P_2 as in Claim 3 (they will be links), and these satisfy (4), too. Let t be a cutvertex of $(G-x)-V(P)$. Since $G-x$ is 2-connected, t is not among the two neighbors of $V(P)$ in $G-x$, and $\{t,x,y\}$ separates G for every $y \in V(P)$.

For the induction step, let G' be the graph obtained from G by contracting an edge $yz \in E(P)$ to a new vertex w, and let P' be the path in G' formed by the edges corresponding to those of P. Clearly, P' is a link in G'. G' is locally connected, since the three neighbors z, x and, say v, of y in G form a path zxv. G' is 3-connected (for otherwise there would be a smallest separator of G containing y, z, which separates $N_G(\{y,z\})$ and, thus, had to contain x; but then $\{x,y\}$ would already be a separator of G, and this is absurd). Furthermore, $G'-x$ is not isomorphic to a cycle, and, thus, contains paths P_1, P_2 in G' satisfying (1)–(4) as in the theorem (with G' for G). Since $(G'-x)-V(P_1)$ is 2-connected, P_1, P_2 do not intersect $V(P) \cup N_{G-x}(V(P))$.

It follows that P_1, P_2 are links in G, too; clearly, $(G-x)-V(P_1)$ and $(G-x)-V(P_2)$ are 2-connected.

This proves that there exists paths P_1, P_2 satisfying (1)–(4), and we prove that for any choice of these (5), (6) are satisfied, too.

To prove (5), consider the graph G' obtained from G by contracting an edge yz, where $y \in V(P_1) \cup V(P_2)$. $N_G(y)$ forms a path txz in G for some $z \in V(P_1) \cup V(P_2)$. Also $N_G(z)$ forms a path of length 2 in G, in which y is an end vertex, so $G(N_G(x)) - y$ is connected. By Corollary 2, G' is locally connected 3-connected.

To prove (6), consider the graph $G' := G - V(P_1)$, and let $N_{G-x}(V(P_1))$ consist of the two vertices t_1, t_2. Since no vertex of P_1 separates $N_G(x)$, $G'(N_{G'}(x))$ remains connected. Since the unique neighbor of t_1 in $V(P_1)$ is an end vertex in $G(N_G(t_1))$, $G'(N_{G'}(t_1))$ is connected and, symmetrically, $G'(N_{G'}(t_2))$ is connected. Since $G(N_G(y)) = G'(N_{G'}(y))$ for all $y \in V(G') - (V(P_1) \cup \{x, t_1, t_2\})$, G' is locally connected. If G' had a separator T of cardinality at most 2 then T would not contain x, since $(G-x)-V(P_1)$ is 2-connected. But then T would separate G, too, since every vertex in $V(P_1)$ is adjacent to x. $\square$

Let us have a look at an example. Take your favourite locally connected 3-connected graph G''. Choose a path $abcd$ of length 3 in G''. Add a second edge from a to b and one from c to d, and subdivide each of the two new edges at least twice. The graph G' obtained in that way contains an a,b-path Q_1 and a c,d-path Q_2 of length at least 3 each, and all the interior vertices of Q_1 and Q_2 have degree 2. To obtain G, add a new vertex x to G' and make it adjacent to every vertex in $V(Q_1) \cup V(Q_2)$. It is not hard to see that G is locally connected 3-connected. Since every neighbor of x in G is adjacent to a neighbor of x of degree 3, x is not incident with a 3-contractible edge. Note that the interior vertices of Q_1, Q_2 induce paths P_1, P_2, respectively, as in Theorem 3. Since G'' has been chosen arbitrarily, we cannot expect much more structure in $G-x$ than provided by Theorem 3.

Theorem 3 has various consequences for locally connected 3-connected graphs, similar to those of the corresponding result on 3-connected graphs [12]. One of them is an alternative proof for Theorem 2, another one will be applied to a coloring problem later:

Corollary 3. *If x is a vertex of degree at most 5 in a locally connected 3-connected graph G then either $G-x$ is a cycle or x is incident with an edge whose contraction yields a locally connected 3-connected graph.*

Proof. Take x as above, assume that $G - x$ is not a cycle, and suppose, to the contrary, that x is not incident with an edge whose contraction yields a locally connected 3-connected graph. Then there exist paths P_1, P_2 of length at least 1 as in Theorem 3. By (1) and (2) of Theorem 3, $X := V(P_1) \cup V(P_2) \cup N_{G-x}(V(P_1))$ is a subset of $N_G(x)$, and by (3), $|X| \geqslant 6$, so $d_G(x) \geqslant |X| \geqslant 6$—a contradiction. $\square$

5. Locally connected 4-connected graphs

In this section, we will characterize the locally connected 4-connected graphs which cannot be transformed to a locally connected 4-connected graph by contracting a single edge.

The *square* G^2 of a graph G is the graph defined by $V(G^2) := V(G)$ and $E(G^2) := \{xy\colon \operatorname{dist}_G(x, y) \in \{1, 2\}\}$, where $\operatorname{dist}_G(x, y)$ denotes the *distance* of x, y in G, i.e. the length of a shortest x, y-path in G ($+\infty$ if x, y are in different components of G). It is easy to see that the square of any graph is locally connected.

The square C_ℓ^2 of a cycle of length $\ell \geqslant 5$ is locally connected 4-connected, and contracting any edge in C_ℓ^2 produces a graph which is not 4-connected. Figure 2 shows an example. As the main theorem of this section indicates, the squares of cycles of length at least 5 are the only locally connected 4-connected graphs which cannot be transformed to a smaller 4-connected graph by contracting a single edge.

We will generalize Propositions 1 and 2 from [16]. To prove that generalization, we need a well-known statement on 2-connected graphs, Theorem 4. Recall that a graph G is called *almost critical*, if it is $\{\emptyset\}$-critical or, equivalently, if G is noncomplete and every fragment of G is intersected by some smallest separator of G. In [18] the following has been proved.

Theorem 4. *Every almost critical graph of connectivity* 2 *has four vertices of degree* 2.

We will use this to prove the following statement.

Lemma 4. *Let x be a vertex of degree* 4 *in a* 4*-connected graph G nonisomorphic to K_5 such that every edge incident with x or incident with some neighbor of x of degree* 4 *is not* 4*-contractible.*

Then every neighbor of x has degree 4*, and $G(N_G(x))$ is one of the graphs $2K_2$, P_4, or C_4, where $2K_2$ denotes the* 1*-regular graph on* 4 *vertices, P_4 denotes the path on* 4 *vertices, and C_4 denotes the cycle on* 4 *vertices.*

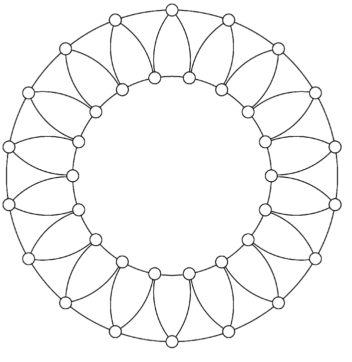

Fig. 2. The square of a cycle of length 36.

Proof. We first need a general statement for vertices w which are not incident with 4-contractible edges of G at all.

Claim 1. *Let $w \in V(G)$ such that every edge incident with w is not 4-contractible. Let F be a T-fragment such that T contains w and a neighbor of w. Then F is intersected by some triangle which contains w and a neighbor of w of degree 4.*

For the proof, let $\mathcal{R} := \{\{w, y\}\colon\ y \in N_G(w)\}$. There exists a $T_B - \mathcal{R}$-end $B \subseteq F$. Let $y \in B \cap N_G(w)$, and let $z' \in T_B \cap N_G(w)$. There exists a $T' \in \mathcal{T}(G)$ containing w, y. By the considerations following (1), T' separates T_B. Since $w \in T' \cap T_B$, there exists a T'-fragment F' such that $F' \cap T_B = \{z\}$ for some z and $|\overline{F'} \cap T_B| \in \{1, 2\}$. If $|B| = 1$ or $F' = \{z\}$ then the assertion is proved by the triangle wyz' or wyz, respectively. So $|B| \geqslant 2$ and F' intersects one of $B, \overline{B}$.

If $F' \cap B \neq \emptyset$ then $|T_G(F', B)| >^* 4$, so $|T_G(F', B) - \{z\}| = |T' - \overline{B}| \geqslant 4$, which is absurd, since T' intersects $\overline{B}$. So $F' \cap B = \emptyset$ and $F' \cap \overline{B} \neq \emptyset$, which implies $B \cap \overline{F'} =^* \emptyset$. Hence $B \subseteq T'$, and from $|B| \geqslant 2$ we deduce $|T_G(F', \overline{B})| = |(T' - B) \cup \{z\}| \leqslant (4 - 2) + 1 < 4$, a contradiction, proving Claim 1.

Now let $\mathcal{S} := \{\{x, y\}\colon\ y \in N_G(x)\}$, where x is the vertex of the statement of our lemma.

Claim 2. *Every $\mathcal{S}$-end has cardinality* 1.

Consider a $T_B - \mathcal{S}$-end B and assume, to the contrary, that it contains more than one vertex. By Claim 1, applied to x, B for w, F, there exists a triangle xyz such that $y \in B$ and one of y, z has degree 4. If $z \in B$ or y had degree 4 then one of $\{z\}, \{y\}$ would be an $\mathcal{S}$-fragment properly contained in B. So $z \in T_B$ and z has degree 4. z must have a neighbor $y' \in B - \{y\}$, for otherwise $B - \{y\}$ would be an $\mathcal{S}$-fragment properly contained in B (as $\{x, y\} \subseteq N_G(B - \{y\}) = (T_B - \{z\}) \cup \{y\}$ would hold), and z must have a neighbor $y'' \in \overline{B}$. By Claim 1, applied to $z, \overline{B}$ for w, F, zxy'' form a triangle.

There exists a T' containing z, y' which must separate the path $y''xy$ induced by $N_G(z) - \{y'\}$. Therefore, $x \in T'$ and $y \in F'$ for some T'-fragment F'. Since $T_G(F', B)$ contains $\{x, z\} \in \mathcal{S}$, $F' \cap B \subsetneq B$ cannot be a $T_G(F', B)$-fragment, and so $|F' \cap T_B| >^* |\overline{B} \cap T'|$ and $|B \cap T'| >^* |\overline{F'} \cap T_B|$. From $y' \in B \cap T'$ and $y'' \in \overline{B} \cap \overline{F'}$ it follows that $|\overline{F'} \cap T_B| \geqslant^* |B \cap T'| \geqslant 1$. As a consequence of $|T' \cap T_B| \geqslant 2$ then $|F' \cap T_B| = |\overline{F'} \cap T_B| = 1$, T' separates T_B and T_B separates T'. However, $|B \cap T'| \geqslant 2$ together with $|T_B \cap T'| \geqslant 2$ is in contradiction with $\overline{B} \cap T' \neq \emptyset$.

Note that, for Claims 1 and 2, we have not used the condition that w or x, respectively, have degree 4.

Claim 3. *If F is a $T - \mathcal{S}$-fragment F such that $F \cap N_G(x) = \{z\}$ then z has degree* 4 *and z has a neighbor in $N_G(x) \cap T$.*

Let B be a $T_B - \mathcal{S}$-end contained in F. Clearly, $\emptyset \neq B \cap N_G(x) \subseteq F \cap N_G(x) = \{z\}$, so $B \cap N_G(x) = \{z\}$ and, since $|B| = 1$, $B = \{z\}$. Since B is an $\mathcal{S}$-fragment, z must have a neighbor y in $N_G(x)$; clearly, $y \notin F \cup \overline{F}$, so $y \in T$, which proves Claim 3.

Claim 4. *Every vertex in $G(N_G(x))$ has degree* 1 *or* 2.

Let $H := G(N_G(x))$. Suppose first that H has a vertex y of degree 3. Then y is contained in every $T \in \mathcal{T}(G)$ containing x, since T must separate H and H is spanned by a star centered at y.

Since $\{x, y\}$ is contained in some $T \in \mathcal{T}(G)$, $G - \{x, y\}$ has connectivity 2, and every fragment of $G - \{x, y\}$ must contain a neighbor of x (and of y). For every neighbor $z \in V(H) - \{y\}$, there exists a $T \in \mathcal{T}(G)$ such that $x, z \in T$. Since $y \in T$, $T - \{x, y\} \in \mathcal{T}(G - \{x, y\})$. Therefore, $G - \{x, y\}$ is an almost critical graph of connectivity 2.

Since the vertices of degree 2 in $G - \{x, y\}$ must be common neighbors of x, y, there are at most 3 of them. This conflicts, however, with Theorem 4. So H has no vertices of degree 3.

Consider an $\mathcal{S}$-end B. By Claim 3, $|B| = 1$, say $B = \{b\}$, and b has a neighbor in $N_G(x)$. Analogously, any $\mathcal{S}$-end C contained in $\overline{B}$ has cardinality 1 and has a neighbor in $N_G(x)$, implying that $|E(H)| \geqslant 2$. Now suppose, to the contrary, that H has an isolated vertex y. Then $H - y$ has a spanning path $y_0 y_1 y_2$. There exists a $T \in \mathcal{T}(G)$ containing x, y_0. Since y_1, y_2 are adjacent, there exists a T-fragment F such that $F \cap N_G(x) = \{y\}$. By Claim 3, y must have a neighbor in H, a contradiction.

This proves Claim 4.

By Claim 4, $G(N_G(x))$ is a graph $2K_2$, a graph P_4, or a graph C_4. In either case, we find a perfect matching $y_0 y_1$, $y_2 y_3$ of $G(N_G(x))$. Take a $T \in \mathcal{T}(G)$ containing x, y_1. Since y_2, y_3 are adjacent, there exists a fragment F such that $F \cap N_G(x) = \{y_0\}$, so y_0 has degree 4 by Claim 3. By symmetry, every vertex in $N_G(x)$ has degree 4. □

Recall that a graph G is *essentially* 4*-edge-connected* if it is 3-edge-connected and for every $X \subseteq E(G)$ with $|X| = 3$, at most one component of $G - X$ has more than one vertex.

Theorem 5. *Let $G \ncong K_5$ be a* 4*-connected graph of minimum degree* 4 *such that every edge incident with a vertex of degree* 4 *is not* 4*-contractible.*

Then G is 4*-regular.*

In particular, G is contraction-critically 4*-connected and therefore, by the results of* [15], *either the square of a cycle of length at least* 6 *or the line graph of a cubic essentially* 4*-edge-connected graph.*

Proof. If G was not 4-regular then there would be a vertex x of degree 4 such that not all of its neighbors had degree 4. Since every edge incident with x is not 4-contractible, this contradicts Lemma 4. □

Equivalently, if a 4-connected graph of minimum degree 4 has a 4-contractible edge then there must be one incident with a degree 4-vertex.

Theorem 5 has an interesting consequence for locally connected 4-connected graphs. For the proof, we recall Theorem 1 of [14].

Theorem 6. [14] *Let A be a $T_A - \mathcal{S}$-atom of a graph G and suppose there exists $S' \in \mathcal{S}$ and $T' \in \mathcal{T}(G)$ such that $S' \subseteq T' - \overline{A}$ and $T' \cap A \neq \emptyset$.*

Then $|A| \leqslant \frac{|T' - T_A|}{2}$.

Theorem 7. *Every locally connected* 4*-connected graph G nonisomorphic to the square of a cycle admits an edge e such that G/e is locally connected* 4*-connected.*

Proof. Let G be a 4-connected graph such that for every 4-contractible edge e, G/e is not locally connected. So, in other words, for every edge $e \in E(G)$, G/e is not locally connected 4-connected. We have to prove that G is the square of a cycle.

Claim 1. *Every edge incident with a vertex of degree* 4 *is not* 4*-contractible.*

For the proof, let x be a vertex of degree 4. If $G(N_G(x))$ has a vertex c of degree 3 then any vertex $y \in N_G(x) - \{c\}$ does not separate $G(N_G(x))$; by Theorem 1, G/xy is locally connected, so G/xy is not 4-connected, implying that there exists a $T \in \mathcal{T}(G)$ containing x, y. Since T separates $N_G(x) - \{y\}$, it must contain c, and Claim 1 is proved in this case.

If, otherwise, $G(N_G(x))$ has maximum degree less than 3 then it possesses a spanning path $P = y_0y_1y_2y_3$; since y_0 does not separate P, G/xy_0 is locally connected. Hence xy_0 is not 4-contractible, implying that there exists a $T \in \mathcal{T}(G)$ containing x, y_0. Since T separates $N_G(x) - \{y_0\}$, it must contain y_2. So the statement of Claim 1 holds for $y \in \{y_0, y_2\}$ and, by symmetry, for $y \in \{y_3, y_1\}$. So in either case, Claim 1 is proved.

In order to apply Theorem 5 we prove that G has a vertex of degree 4. Let $\mathcal{S} := \{\{x, y\}\colon\ y \in N_G(x),\ G(N_G(x)) - y$ is connected$\}$. By assumption, every $S \in \mathcal{S}$ is contained in some $T \in \mathcal{T}(G)$, and every $x \in V(G)$ is contained in two distinct members $\{x, y_1\}$, $\{x, y_2\}$ of $\mathcal{S}$ (where y_1, y_2 may be chosen as two distinct end vertices of a spanning tree of $G(N_G(x))$).

In particular, G is $\mathcal{S}$-critical. By Theorem 6, it admits a $T_A - \mathcal{S}$-atom A with $|A| \leqslant \kappa(G)/2 = 2$. We prove $|A| = 1$. For suppose, to the contrary, that $A = \{x, y\}$ for distinct x, y. There exists an $S \in \mathcal{S} - \{\{x, y\}\}$ containing x. Clearly, $S \cap T_A \neq \emptyset$, and we find a $T \in \mathcal{T}(G)$ containing S. By Theorem 6, $|A| \leqslant |T_A - T|/2 < 2$, a contradiction.

Hence G satisfies the conditions in Theorem 5. Since a connected cubic graph G nonisomorphic to K_4 has an edge not contained in a triangle, its line graph $L(G)$ is locally disconnected. Therefore, G is the square of a cycle of length at least 5. (Note that $L(K_4) \cong C_6^2$.) □

It is easy to see that for $\ell \geqslant 7$, C_ℓ^2 can be transformed to $C_{\ell-2}^2$ by contracting two edges. Taking $C_5^2 \cong K_5$ and $C_6^2 \cong K_{2,2,2} \cong L(K_4)$ into account, we thus obtain the following corollary.

Corollary 4. *Every locally connected* 4*-connected graph G nonisomorphic to K_5 and $K_{2,2,2}$ has a locally connected* 4*-connected minor H such that* $0 < |V(G)| - |V(H)| < 3$.

6. Higher connectivity

For $k \geqslant 5$, the class of locally connected 5-connected graphs without 5-contractible edges is rich, as it is demonstrated by the following example.

For $\ell \geqslant 3$ consider the wheel $W_{2\ell}$, where the central vertex x has degree 2ℓ and the border vertices are denoted by $x_0, \ldots, x_{2\ell-1}$. Let H_ℓ be the graph obtained from this wheel by adding 2ℓ new vertices $y_0, \ldots, y_{2\ell-1}$, where we connect y_{2i} to y_{2i+1}, x_{2i}, and x_{2i+1}, and connect y_{2i+1} to y_{2i}, x_{2i+1}, and x_{2i+2} (indices modulo 2ℓ). We refer to the pair (y_{2i}, y_{2i+1}) as to a *spoke* of H_ℓ. Figure 3 displays an example with six spokes.

Now let H be an arbitrary 3-edge-connected multigraph. For each vertex x of degree d in H, we take a copy H_x of H_d and a bijection ϕ_x from the edges incident with x to the *spokes* of H_x. We may choose the H_x being vertex disjoint. Now, for each edge $e = xy$ in H we consider the two spokes $\phi_x(e) = (a, b)$ and $\phi_y(e) = (c, d)$ and identify either $a = c$ and $b = d$ or $a = d$ and $b = c$. The graph G obtained in this way is essentially 6-connected, i.e. all separators but those

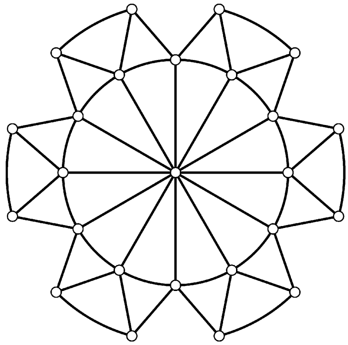

Fig. 3. The graph H_6.

consisting of the neighborhood of some vertex (the *trivial separators*) have at least 6 vertices. Furthermore, it cannot be contracted to a 5-connected graph by contracting less than 4 edges. The class of graphs obtained in that way contains infinitely many planar graphs, and also graphs with arbitrarily large complete minors.

As these examples suggest, it is difficult to determine the minor minimal 5-connected graphs in form of a list. Indeed, this is an open problem:

Conjecture 1. [7] *Every* 5*-connected graph contains a minor isomorphic to one of the six graphs* K_6, $K_{2,2,2,1}$, $C_5 + \overline{K_3}$, I, $\tilde{I}$, *or* G_0.

Here I *denotes the icosahedron,* $\tilde{I}$ *is the graph obtained from* I *by replacing the edges of a cycle* $abcdea$ *induced by the neighborhood of some vertex with the edges of a cycle* $abceda$, *and* G_0 *is the graph obtained from the icosahedron by deleting a vertex* w, *replacing the edge* ab *of a cycle* $abcdea$ *induced by the neighborhood of* w *with the two edges* ac *and* ad, *and, finally, identifying* b *and* e.

Dirac showed that every 5-connected planar graph contains the icosahedron as a minor [5]. Recently, Fijavž generalized this by determining the four minor minimal 5-connected projective planar graphs to be the four projective planar ones mentioned in Conjecture 1 [7], which are K_6, I, $\tilde{I}$, and G_0.

A slightly different conjecture appeared in [12].

Conjecture 2. [12] *There exist integers* b, h *such that every* 5*-connected graph* G *on at least* b *vertices has a* 5*-connected minor* H *such that* $0 < |V(G)| - |V(H)| < h$.

The corresponding statement does hold if 5 is replaced by some integer $k < 5$, and it does not hold if 5 is replaced by some integer $k > 5$, so here is a rather interesting gap to close. In [11] it has been proved that if G is a 5-connected graph on at least 12 vertices such that for every $T \subseteq V(G)$ with $|T| = 5$, at most one component of $G - T$ has more than one vertex, then G can be transformed to a 5-connected graph by contracting 1, 2, 3, or 4 edges.

7. Cycle double covers in locally connected graphs

In this section we apply Corollary 1 in order to prove that every bridgeless locally connected graph admits a cycle double cover.

Recall that a graph H is called *even* if every vertex in H has even degree. A system $(H_1, \dots, H_k)$ of k even subgraphs of some graph G is called a *k-cycle-double-cover* or, briefly, a *cycle double cover* of G, if every edge of G is contained in exactly two of the subgraphs H_i (formally, $|\{i \in \{1, \dots, k\}: e \in E(H_i)\}| = 2$ for every $e \in E(G)$).

The famous cycle double cover conjecture (CDCC) states that every bridgeless graph admits a cycle double cover [22,23]. Here we prove this in the case of locally connected graphs, using a contraction argument.

Theorem 8. *Every bridgeless locally connected graph admits a* 3*-cycle-double-cover.*

Proof. It suffices to prove the statement for every connected locally connected graph G on at least 3 vertices. We prove this by induction on $|V(G)|$. For $|V(G)| = 3$, the statement is trivial, so let $|V(G)| > 3$. By Corollary 1, there exists an edge xy such that the graph G' obtained from G by contracting xy to a new vertex w remains connected locally connected and, thus, admits a 3-cycle-double-cover (H_1', H_2', H_3') by induction. From local connectivity we deduce that there exists a vertex z in G such that xyz forms a triangle in G. We may assume that the edge wz of G' is contained in H_1', H_2' without loss of generality.

Now we construct subgraphs H_1, H_2, H_3 of G from H_1', H_2', H_3' as follows, starting with three empty graphs. If an edge $e' \in E(H_i')$ corresponds to exactly one edge e in $E(G)$ then we add e to $E(H_i)$. If, otherwise, $e' = wz'$ in $E(H_i') \cap E(H_j')$, $i \neq j$ corresponds to the two edges xz' and yz' then we proceed as follows: If $z \neq z'$ then we let k be the element in $\{1, 2, 3\}$ distinct from i, j and add xz' to both $E(H_i)$, $E(H_k)$ and add yz' to both $E(H_j)$, $E(H_k)$.

This construction leads to a collection H_1, H_2, H_3 of subgraphs of G such that every edge in $E(G)$ distinct from xy, xz, yz is contained in exactly two of them, xy, xz, yz are contained in none of them, all vertices distinct from x, y, z have even degree in each of the subgraphs H_1, H_2, H_3, z has odd degree in both H_1, H_2 and even degree in H_3, the sum of the degrees of x, y is odd in H_1, H_2, and even in H_3. The parities of the degree of x in H_1 and in H_2 fix the parity of the degree of x in H_3 and, thus, the parities of the degrees of y in either of H_1, H_2, H_3. In either case it is possible to add each of the three edges xy, xz, yz to exactly two of the H_i in order to construct even subgraphs, as it demonstrated in Table 1. □

As K_2 is the only connected locally connected graph which has bridge, Theorem 8 implies immediately the formally stronger statement that every locally connected graph without a component isomorphic to K_2 has a 3-cycle-double-cover.

Another class of locally connected graphs arises from a product construction. The *lexicographic product* $G[H]$ of two graphs G, H is defined by $V(G[H]) = V(G) \times V(H)$ and $E(G[H]) := \{(x_1, y_1)(x_2, y_2): x_1x_2 \in E(G) \text{ or } (x_1 = x_2 \in V(G) \wedge y_1y_2 \in E(H))\}$.

It is easy to see that if G is a connected graph on at least 2 vertices and $\ell \geqslant 2$ then $G[K_\ell]$ is a locally connected bridgeless graph. So, by Theorem 8, $G[K_\ell]$ has a 3-cycle-double-cover.

Table 1

$d_{H_1}(x)$	$d_{H_2}(x)$	$d_{H_3}(x)$	$d_{H_1}(y)$	$d_{H_2}(y)$	$d_{H_3}(y)$	H_1	H_2	H_3
Odd	Odd	Even	Even	Even	Even	xz	xy, yz	xy, xz, yz
Odd	Even	Odd	Even	Odd	Odd	xy, yz	xy, xz	xz, yz
Even	Odd	Odd	Odd	Even	Odd	xy, xz	xy, yz	xz, yz
Even	Even	Even	Odd	Odd	Even	yz	xy, xz	xy, xz, yz

Also squares of bridgeless graphs are locally connected and bridgeless, hence they admit 3-cycle-double-covers, too.

In fact, every graph obtained from a graph having a 3-cycle-double-cover by performing a nonproper splitting at some vertex admits a 3-cycle-double-cover. The same (constructive) proof technique thus leads to other classes of graphs having a 3-cycle-double-cover, for example, to the following which we give without proof.

Theorem 9. *Every bridgeless graph of order $n \geqslant 3$ admitting a system of $n-2$ edge disjoint triangles has a 3-cycle-double-cover.*

8. Cyclic colorings of locally connected planar graphs

Let G be a 3-connected planar graph G. A mapping $f: V(G) \to C$ is called a *cyclic C-coloring* or a *cyclic $|C|$-coloring* or, more briefly, a *cyclic coloring* of G, if $f(x) \neq f(y)$ for any two distinct vertices x, y belonging to the same facial cycle. So the *cyclic chromatic number* $\chi_c(G) := \min(\{k\colon$ there exists a cyclic k-coloring$\})$ is the smallest number of colors needed to color the vertices of G such that distinct vertices on the same face receive distinct colors. A trivial lower bound for $\chi_c(G)$ is $\Delta^*(G)$, the maximum degree of the geometric dual of G.

Plummer and Toft conjectured $\chi_c(G) \leqslant \Delta^*(G)+2$ for every planar 3-connected graph [19]. For $\Delta^*(G)=3$, the conjecture follows from the 5-color-theorem, for $\Delta^*(G)=4$, it has been proved in [2]. It is true for $\Delta^*(G) \geqslant 24$ [9], and it is even possible to improve the bound for graphs with large facial cycles: For every planar 3-connected graph G with $\Delta^*(G) \geqslant 60$, even $\chi_c(G) \leqslant \Delta^*(G)+1$ holds [6].

Here we will sketch a proof for the locally connected case, which follows mostly the lines of [19]—with two interesting short cuts relying on Corollary 3 and on the absence of certain face patterns in locally connected 3-connected planar graphs.

Before starting this, we consider graphs without large facial cycles and obtain the following corollary of the 4-color-theorem [20].

Theorem 10. *Let G be a 3-connected planar graph such that $\Delta^*(G) \leqslant 5$ and such that the facial cycles of length exceeding 3 are vertex disjoint. Then G is locally connected and $\chi_c(G) \leqslant \Delta^*(G)+2$.*

Proof. As any vertex is contained in at most one facial cycle of length exceeding 3, its neighborhood induces a path or a cycle. Therefore, G is locally connected. It remains to prove that $\chi_c(G) \leqslant \Delta^*(G)+2$.

For $i \in \{4,5\}$, let $\mathcal{C}_i$ denote the set of facial cycles of length i in G. If $\Delta^*(G)=3$ then the assertion is true by Heawood's 5-color theorem (see [4]). If $\Delta^*(G)=4$ then it follows from the results in [2].

Suppose that $\Delta^*(G)=5$. We have to prove that G admits a cyclic 7-coloring. Let G' be the planar 3-connected graph by adding an arbitrary chord to each member of $\mathcal{C}_4 \cup \mathcal{C}_5$. By the 4-color-theorem there exists a $\{5,6,7,4\}$-coloring f of G'. Observe that f is also a coloring of G such that each member of $\mathcal{C}_4 \cup \mathcal{C}_5$ receives at least 3 colors. Therefore, for every $C \in \mathcal{C}_4$ we can choose a representative vertex $x_C \in V(C)$ such that $f(x_C) \in \{5,6\}$. Furthermore, every $C \in \mathcal{C}_5$ has two distinct nonadjacent vertices y_C, z_C such that $f(y_C) \neq f(z_C)$ and $f(y_C), f(z_C) \in \{5,6,7\}$, as at most 2 vertices in C received color 4. Let $X := \{x_C\colon C \in \mathcal{C}_4\} \cup \{y_C, z_C\colon C \in \mathcal{C}_5\}$. Then $V(C) \cap X = \{x_C\}$ if $C \in \mathcal{C}_4$ and $V(C) \cap X = \{y_C, z_C\}$ if $C \in \mathcal{C}_5$. Now we can construct a triangulation G'' from G by adding a chord to each cycle $C \in \mathcal{C}_4$ and by adding a pair of chords to each cycle $C \in \mathcal{C}_5$ in such a way that none of these chords contains a vertex from X. By the 4-color-theorem, G'' admits a $\{1,2,3,4\}$-coloring g, which is also a coloring of G such that the three vertices in each $C - X$, $C \in \mathcal{C}_4 \cup \mathcal{C}_5$, receive different colors. Now it is easy to check that $h(x) := f(x)$ for $x \in X$ and $h(x) := g(x)$ for $x \in V(G) - X$ defines a cyclic $\{1,2,3,4,5,6,7\}$-coloring of G. $\square$

Now we are prepared to prove the main result of this section.

Theorem 11. *For every locally connected* 3*-connected planar graph* G*,* $\chi_c(G) \leqslant \Delta^*(G) + 2$ *holds.*

Sketch of proof. First note that the statement is true for a wheel W_n on $n+1$ vertices, as $\chi_c(W_n) = n + 1 \leqslant n + 2 = \Delta^*(W_n) + 2$. We proceed by induction on $|V(G)|$. The induction starts since the smallest locally connected 3-connected graph K_4 is isomorphic to the wheel W_3.

If there is a separating triangle T then we consider a component C of $G - T$ and apply induction to the locally connected 3-connected planar induced subgraphs $G_1 := G(V(C) \cup T)$ and $G_2 := G - V(C)$. After relabelling colors we thus find cyclic $\{1,\dots,k\}$-colorings f_1, f_2 of G_1, G_2, respectively, such that $f_1|T = f_2|T$ and $k = \max\{\Delta^*(G_1)+2, \Delta^*(G_2)+2\}$ holds. Since the facial cycles of G are precisely the facial cycles of G_1 and G_2 distinct from T, $k = \Delta^*(G)+2$ follows, and $f(x) := f_i(x)$ if $x \in V(G_i)$ for $i \in \{1,2\}$ defines a cyclic k-coloring of G, thus proving the assertion.

Hence we may assume that G admits no separating triangle. Let us enumerate the neighbors of some vertex x of G according to the rotation scheme of an embedding of G. Since G has no separating triangles, adjacent vertices in $N_G(x)$ must be consecutive in this order. It follows that $H := G(N_G(x))$ is a connected graph of maximum degree at most 2 and, thus, a path or a cycle. From this it follows, in particular, that x is incident with at least $d_G(x) - 1$ facial triangles, and so facial cycles of length exceeding 4 are vertex disjoint.

Let $F(G)$ denote the set of all facial cycles of G and define $\mathcal{C}(x) := \{C \in F(G)\colon x \in V(C)\}$. Consider the Euler *distribution* Φ defined by $\Phi(x) := 1 - (d_G(x))/2 + \sum_{C \in \mathcal{C}(x)} \frac{1}{|V(C)|}$ for $x \in V(G)$, and observe that

$$\sum_{x \in V(G)} \sum_{C \in \mathcal{C}(x)} \frac{1}{|V(C)|} = \sum_{C \in F(G)} \sum_{x \in V(C)} \frac{1}{|V(C)|} = \sum_{C \in F(G)} 1 = \bigl|F(G)\bigr|.$$

By Euler's formula, $\sum_{x \in V(G)} \Phi(x) = |V(G)| - |E(G)| + |F(G)| = 2$. Hence there exists a vertex x of degree at most 5 in G such that $\Phi(x) > 0$ (see [19]).

If $G(N_G(x))$ is a cycle then by Corollary 3 there exists an edge xy such that the graph G' obtained from contracting xy to a single vertex w is locally connected 3-connected. If, otherwise, $G(N_G(x))$ is a path then we take, more specifically, one of its end vertices, say y, and the graph G' obtained from contracting xy to a single vertex w will be locally connected 3-connected by Corollary 2.

By choice of xy, x is contained in $d_G(x) - 1$ facial triangles and one further facial cycle C of length $\ell \geqslant 3$ which contains y as well.

By induction, there exists a cyclic $\{1,\dots,k\}$-coloring f of G', where $k := \Delta^*(G') + 2$. The partial coloring defined by $g(z) := f(z)$ for $z \in V(G) - \{x,y\}$ and $g(y) := f(w)$ assigns $\ell - 1$ different colors to $C - x$ and at most $d_G(x) - 2 \leqslant 3$ further colors to the vertices in $N_G(x) - V(C)$. So the facial cycles of G incident with x receive at most $\ell - 1 + d_G(x) - 2 \leqslant$

$k = \Delta^*(G') + 2 \leqslant \Delta^*(G) + 2$ different colors, implying that we may assign $g(x) := c$ for an appropriate color c of $\{1, \ldots, \Delta^*(G)+2\}$ in order to construct a cyclic $\{1, \ldots, \Delta^*(G)+2\}$-coloring of G unless $\ell - 3 + d_G(x) = k = \Delta^*(G') + 2 = \Delta^*(G) + 2$.

The latter equalities imply $d_G(x) = 5$ and $\ell = \Delta^*(G)$, and from $\Phi(x) > 0$ we deduce $\ell \in \{3, 4, 5\}$ easily (or with help of the table in Section 2 of [19]). So $\Delta^*(G) \leqslant 5$ as well, implying that G is a graph satisfying the conditions of Theorem 10. Consequently, $\chi_c(G) \leqslant \Delta^*(G) + 2$. $\square$

As the wheels show, we may not expect a bound for $\chi_c(G)$ better than $\Delta^*(G) + 1$.

9. Open problems

To continue the investigations in the latter chapter, note that there exist infinitely many 3-connected graphs G with $\chi_c(G) = \Delta^*(G) + 2$ (see [19]). However, none of these graphs is of the type considered in Theorems 10 and 11. So there is still a gap to fill in the locally connected case:

Conjecture 3. *If G is a locally connected 3-connected planar graph then $\chi_c(G) \leqslant \Delta^*(G) + 1$ holds.*

A two-paths-theorem similar to Theorem 3 has been designed for 3-connected, not necessarily locally connected graphs in [13], in order to partially answer the following conjecture on 3-connected graphs, due to McCuaig and Ota.

Conjecture 4. [17] *For every integer $k > 0$ there exists a least integer $f(k)$ such that every 3-connected graph G on at least $f(k)$ vertices has a connected subgraph H on exactly k vertices such that $G - V(H)$ is 2-connected.*

It might be possible to use Theorem 3 for proving this conjecture restricted to locally connected graphs. This is also supported by the observation that the known graphs from which good nontrivial lower bounds for f in Conjecture 4 can be derived are triangle free, whereas locally connected graphs have many triangles.

Let us have a look at another attractive question, brought into play by Ryjáček.

The *girth* $g(G)$ of a graph G nonisomorphic to a forest is the length of a shortest cycle, and its *circumference* $c(G)$ is the length of a longest cycle. We call a graph G *weakly pancyclic*, if it is either a forest or it is not a forest and contains a cycle of length ℓ for every integer ℓ with $g(G) \leqslant \ell \leqslant c(G)$. Due to a beautiful simple proof by Balister, every spheric triangulation (i.e. maximally planar graph) is weakly pancyclic (see [21]). Ryjáček conjectured the following generalization.

Conjecture 5. [21] *Every locally connected graph is weakly pancyclic.*

Before reporting on further partial results on this conjecture, let us consider an example showing that we may not expect hamiltonicity in Conjecture 5 even when strengthening the conditions to both local and global connectivity arbitrarily.

We call a graph G *locally k-connected*, if $G(N_G(x))$ is k-connected for every vertex $x \in V(G)$. Consider a vertex x in a smallest separating set T of a locally k-connected connected graph G and two neighbors $y, z \in N_G(x)$ in distinct components of $G - T$; $G(N_G(x))$ contains k openly disjoint y,z-paths, and each of them must intersect T in an inner vertex, implying $|T| \geqslant k + 1$. So every locally k-connected connected graph is already $(k + 1)$-connected [3].

A large class of locally k-connected graphs can be obtained as follows. Let G be a graph and $\ell \geqslant 2$. For a vertex (x_1, x_2) of the lexicographic product $G[K_\ell]$, consider the subgraph $H := N_{G[K_\ell]}((x_1, x_2))$ induced by its neighborhood. Since (x_1, x_2') is the center of a spanning star of H for every $x_2' \in V(K_\ell) - \{x_2\}$, H is $(\ell - 1)$-connected, so $G[K_\ell]$ is locally $(\ell - 1)$-connected. Moreover, if G is k-connected then $G[K_\ell]$ is $(k \cdot \ell)$-connected. Given $k, \ell \geqslant 1$ arbitrarily and $b > k \cdot \ell$, the graphs $K_{k,b}[K_\ell]$ show that there are nonhamiltonian graphs of arbitrarily large local connectivity and arbitrarily large connectivity.

In [10], it has, however, been proved that $G[K_2]$ is weakly pancyclic for every connected graph G, and in [8], squares of connected graphs turned out to be weakly pancyclic. Furthermore, it is not hard to prove that every connected locally connected graph G of maximum degree at most 4 nonisomorphic to the square of a cycle has a vertex x such that $G - x$ is connected locally connected. As it has been proved in [3], every connected locally connected graph of maximum degree 4 nonisomorphic to $K_{1,1,3}$ is hamiltonian. These facts lead to an easy induction proof for the statement of Conjecture 5 restricted to graphs of maximum degree at most 4.

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, MacMillan, 1976.
[2] O. Borodin, A new proof of the 6 color theorem, J. Graph Theory 19 (4) (1995) 507–521.
[3] G. Chartrand, R.E. Pippert, Locally connected graphs, Časopis Pěst. Mat. 99 (1974) 158–163.
[4] R. Diestel, Graph Theory, Grad. Texts in Math., vol. 173, Springer, 1997.
[5] G.A. Dirac, On the four-colour conjecture, Proc. London Math. Soc. (3) 13 (1963) 193–218.
[6] H. Enomoto, M. Horňák, S. Jendrol', Cyclic chromatic number of 3-connected plane graphs, SIAM J. Discrete Math. 14 (2001) 121–137.
[7] G. Fijavž, Minor minimal 5-connected projective planar graphs, preprint from the author, Ljubljana, 2005, submitted for publication.
[8] H. Fleischner, In the square of graphs, Hamiltonicity and pancyclicity, Hamiltonian connectedness and panconnectedness are equivalent concepts, Monatsh. Math. 82 (2) (1976) 125–149.
[9] M. Horňák, S. Jendrol', On a conjecture by Plummer and Toft, J. Graph Theory 30 (3) (1999) 177–189.
[10] T. Kaiser, M. Kriesell, Pancyclicity of lexicographical products, Graphs Combin., in press.
[11] M. Kriesell, How to contract an essentially 6-connected graph to a 5-connected graph, Discrete Math., in press.
[12] M. Kriesell, A survey on contractible edges in graphs of a prescribed vertex connectivity, Graphs Combin. 18 (1) (2002) 1–30.
[13] M. Kriesell, Contractible subgraphs in 3-connected graphs, J. Combin. Theory Ser. B 80 (1) (2000) 32–48.
[14] W. Mader, Generalizations of critical connectivity of graphs, Discrete Math. 72 (1988) 267–283.
[15] N. Martinov, A recursive characterization of the 4-connected graphs, Discrete Math. 84 (1) (1990) 105–108.
[16] N. Martinov, Uncontractible 4-connected graphs, J. Graph Theory 6 (3) (1982) 343–344.
[17] W. McCuaig, K. Ota, Contractible triples in 3-connected graphs, J. Combin. Theory Ser. B 60 (2) (1994) 308–314.
[18] L. Nebeský, On induced subgraphs of a block, J. Graph Theory 1 (1) (1977) 69–74.
[19] M.D. Plummer, B. Toft, Cyclic coloration of 3-polytopes, J. Graph Theory 11 (4) (1987) 507–515.
[20] N. Robertson, D. Sanders, P. Seymour, R. Thomas, The four-colour theorem, J. Combin. Theory Ser. B 70 (1) (1997) 2–44.

[21] Z. Ryjáček, Weak pancyclicity of locally connected graphs, Problem 416, Discrete Math. 272 (2003) 305–306.
[22] P.D. Seymour, Sums of circuits, in: Graph Theory and Related Topics, Proc. Conf., Univ. Waterloo, Waterloo, Ontario, 1977, Academic Press, New York, 1979, pp. 341–355.
[23] G. Szekeres, Polyhedral decompositions of cubic graphs, Bull. Austral. Math. Soc. 8 (1973) 367–387.
[24] C. Thomassen, Nonseparating cycles in k-connected graphs, J. Graph Theory 5 (1981) 351–354.
[25] C. Thomassen, Kuratowski's theorem, J. Graph Theory 5 (1981) 225–241.